Particle Swarm Optimization: A Universal Optimizer?

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Presentation Outline I

Introduction

Original PSO has been developed to solve optimization problems that are

- unconstrained/boundary constrained
- static
- single-objective
- continuous-valued

However:

- Can PSO be used to solve optimization problems of different classes, without significantly changing the principles of the basic PSO?
- If this is the case, we say that PSO is a universal optimizer
- For each problem class, what are the issues, and how can PSO be adapted to address these issues, while still maintaining the behavioral principles of PSO?

Goal:

- To show that PSO is a universal optimizer
- Not to present a review of the best possible approaches to solve optimization problems of the different problem classes, but to show that PSO can solve these problems
- Focus is on simple, efficient approaches

Optimization Problem Classes

A number of different optimization problem classes can be identified

- Unconstrained
- Boundary constrained
- Constrained
- Multi-objective, many-objective
- Multi-modal
- Dynamic and noisy
- Continuous-valued versus discrete-valued
- Large scale problems

What are the main components?

- a swarm of particles
- each particle represents a candidate solution
- elements of a particle represent parameters to be optimized

The search process:

- Position updates

$$\mathbf{x}_i(t+1) = \mathbf{x}_i(t) + \mathbf{v}_i(t+1), \quad \mathbf{x}_{ij}(0) \sim U(x_{min,j}, x_{max,j})$$

- Velocity (step size)
  - drives the optimization process
  - step size
  - reflects experiential knowledge and socially exchanged information

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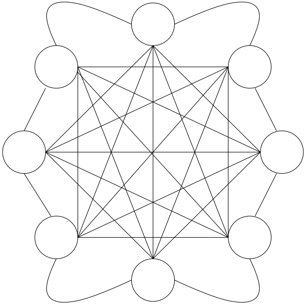
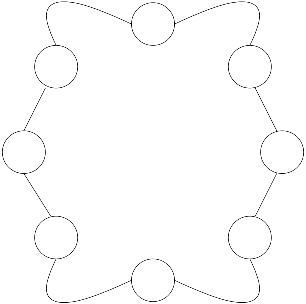
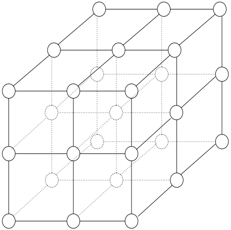
Social network structures are used to determine best positions/attractors



: Star Topology



: Ring Topology



: Von Neumann Topology

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- uses the star social network
- velocity update per dimension:

$$v_{ij}(t+1) = v_{ij}(t) + c_1 r_{1j}(t)[y_{ij}(t) - x_{ij}(t)] + c_2 r_{2j}(t)[\hat{y}_j(t) - x_{ij}(t)]$$

- $v_{ij}(0) = 0$ (preferred)
- c_1, c_2 are positive acceleration coefficients
- $r_{1j}(t), r_{2j}(t) \sim U(0,1)$
- note that a random number is sampled for each dimension

- $\mathbf{y}_i(t)$ is the personal best position calculated as (assuming minimization)

$$\mathbf{y}_i(t+1) = \begin{cases} \mathbf{y}_i(t) & \text{if } f(\mathbf{x}_i(t+1)) \geq f(\mathbf{y}_i(t)) \\ \mathbf{x}_i(t+1) & \text{if } f(\mathbf{x}_i(t+1)) < f(\mathbf{y}_i(t)) \end{cases}$$

- $\hat{\mathbf{y}}(t)$ is the global best position calculated as

$$\hat{\mathbf{y}}(t) \in \{\mathbf{y}_0(t), \ldots, \mathbf{y}_{n_s}(t)\} | f(\hat{\mathbf{y}}(t)) = \min\{f(\mathbf{y}_0(t)), \ldots, f(\mathbf{y}_{n_s}(t))\}$$

or (removing memory of best positions)

$$\hat{\mathbf{y}}(t) = \min\{f(\mathbf{x}_0(t)), \ldots, f(\mathbf{x}_{n_s}(t))\}$$

where n_s is the number of particles in the swarm

- uses the ring social network

$$v_{ij}(t+1) = v_{ij}(t) + c_1 r_{1j}(t)[y_{ij}(t) - x_{ij}(t)] + c_2 r_{2j}(t)[\hat{y}_{ij}(t) - x_{ij}(t)]$$

- $\hat{\mathbf{y}}_i$ is the neighborhood best, defined as

$$\hat{\mathbf{y}}_i(t+1) \in \{\mathcal{N}_i | f(\hat{\mathbf{y}}_i(t+1)) = \min\{f(\mathbf{x})\}, \ \forall \mathbf{x} \in \mathcal{N}_i\}$$

  with the neighborhood defined as

$$\mathcal{N}_i = \{\mathbf{y}_{i-n_{\mathcal{N}_i}}(t), \mathbf{y}_{i-n_{\mathcal{N}_i}+1}(t), \ldots, \mathbf{y}_{i-1}(t), \mathbf{y}_i(t), \mathbf{y}_{i+1}(t), \ldots, \mathbf{y}_{i+n_{\mathcal{N}_i}}(t)\}$$

  where $n_{\mathcal{N}_i}$ is the neighborhood size
- neighborhoods based on particle indices, not spatial information
- neighborhoods overlap to facilitate information exchange

- previous velocity, $\mathbf{v}_i(t)$
  - inertia component
  - memory of previous flight direction
  - prevents particle from drastically changing direction
- cognitive component, $c_1\mathbf{r}_1(\mathbf{y}_i - \mathbf{x}_i)$
  - quantifies performance relative to past performances
  - memory of previous best position
  - nostalgia
- social component, $c_2\mathbf{r}_2(\hat{\mathbf{y}}_i - \mathbf{x}_i)$
  - quantifies performance relative to neighbors
  - envy

Synchronous Iteration Strategy

```
Create and initialize the swarm;
repeat
    for each particle do
        Evaluate particle's fitness;
        Update particle's personal best position;
        Update particle's neighborhood best position;
    end
    for each particle do
        Update particle's velocity;
        Update particle's position;
    end
until stopping condition is true;
```

Asynchronous Iteration Strategy

```
Create and initialize the swarm;
repeat
    for each particle do
        Update the particle's velocity;
        Update the particle's position;
        Evaluate particle's fitness;
        Update the particle's personal best position;
        Update the particle's neighborhood best position;
    end
until stopping condition is true;
```

What is the problem?

- PSO was originally developed for optimizing continuous-valued variables
- That is $x_{ij} \in \mathbb{R}$
- Uses vector algebra on floating-point vectors to adjust search positions

How do we adapt PSO so that $x_{ij} \in \{0, 1\}$?

Adapting PSO for binary-valued variables: Binary PSO

- Velocity remains a floating-point vector, but meaning changes
- Velocity is no longer a step size, but is used to determine a probability of selecting bit 0 or bit 1
- Position is a bit vector, i.e. $x_{ij} \in \{0, 1\}$
- How to interpret velocity as a probability?

$$p_{ij}(t) = \frac{1}{1 + e^{-v_{ij}(t)}}$$

- Then, position update changes to

$$x_{ij}(t+1) = \begin{cases} 1 & \text{if } U(0,1) < p_{ij}(t+1)) \\ 0 & \text{otherwise} \end{cases}$$

Velocity clamping:

- sets the minimal probability for a bit change
- if $V_{max,j} = 4$, then $\text{sig}(V_{max,j}) = 0.982$ is the probability of x_{ij} to change to bit 1, and 0.018 the probability to change to bit 0
- small values for $V_{max,j}$ promotes exploration
- for $V_{max,j} = 0$, the search changes to a random search
- large values for $V_{max,j}$ promotes exploitation
- start with small $V_{max,j}$ that increases over time

Inertia weight:

- $w < 1$ works against convergence, as v_{ij} becomes zero over time, and each bit then has a 50% change of changing
- velocity should not become zero
- start with small w, increase over time

Velocity initialization: Initialize to zero.

- for $v_{ij} > 1$, $\lim_{t\to\infty} \text{sig}(v_{ij}(t)) \to 1$ and the probability that all bits change to 1 increases
- for $v_{ij} < -1$, $\lim_{t\to\infty} \text{sig}(v_{ij}(t)) \to 0$ and the probability that all bits change to 0 increases

Some issues with the binary PSO:

- Changes the meaning of the velocity update
  - No longer a step size
  - No longer a search trajectory
- Effect of control parameters change
- Theoretical analysis of standard PSO no longer applies

An approach to solve a $\mathbb{B}^{n_x}$-dimensional problem in $\mathbb{R}^4$

- Velocities and particle positions remain floating-point vectors
- Find a bitstring generating function, used to generate the bitstring solution
- The generating function:

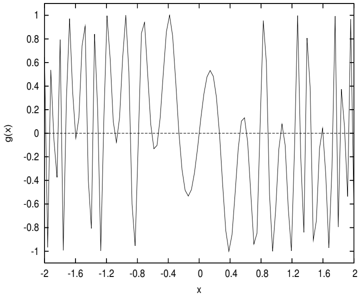
$$g(x) = \sin(2\pi(x - a) \times b \times \cos(2\pi(x - a) \times c)) + d$$

where x is a single element from a set of evenly separated intervals determined by the required number of bits that need to be generated

$$g(x) = \sin(2\pi(x - a) \times b \times \cos(2\pi(x - a) \times c)) + d$$

The coefficients determine the shape of the generating function:

- *a*: horizontal shift of generating function
- *b*: maximum frequency of the sin function
- *c*: frequency of the cos function
- *d*: vertical shift of generating function



Use a standard PSO to find the best values for these coefficients

Generate a swarm of 4-dimensional particles;
repeat
  Apply any PSO for one iteration;
  for *each particle* **do**
    Substitute values for coefficients a, b, c and d into generating function;
    Produce n_x bit-values to form a bit-vector solution;
    Calculate the fitness of the bit-vector solution in the original bit-valued space;
  end
until *a convergence criterion is satisfied*;

Assuming minimization,

Boundary constrained optimization problem:

$$\begin{array}{rll} \text{minimize} & f(\mathbf{x}), & \mathbf{x} = (x_1, x_2, \ldots, x_{n_x}) \\ \text{subject to} & & x_j \in \text{dom}(x_j) \end{array}$$

where $\mathbf{x} \in \mathcal{F} = \mathcal{S}$, and $\text{dom}(x_j)$ is the domain of variable x_j.

Multi-solution problem: Find a set of solutions,

$$\mathcal{X} = \{\mathbf{x}_1^*, \mathbf{x}_2^*, \ldots, \mathbf{x}_{n_\mathcal{X}}^*\}$$

such that each $\mathbf{x}^* \in \mathcal{X}$ is a minimum of the general optimization problem

Niching capability of PSO:

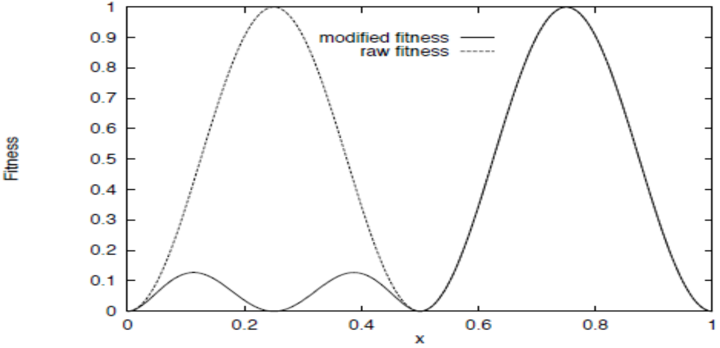
- Can the *gbest* PSO find more than one solution?
  - Formal proofs showed that all particles converge to a weighted average of their personal best and global best positions

$$\lim_{t\to\infty} \mathbf{x}_i(t) = \frac{c_1\mathbf{y}_i + c_2\hat{\mathbf{y}}}{c_1 + c_2}$$

  - Therefore, only one solution can be found
  - What if we re-run the algorithm? No guarantee to find different solutions
- What about *lbest* PSO?
  - Neighborhoods may converge to different solutions
  - However, due to overlapping neighborhoods, particles are still attracted to one solution
  - Formal proof exist to show that all particles converge in the limit

Sequential niching, stretching the function to remove found minima

Create and initialize a n_x-dimensional swarm, S, and $\mathcal{X} = \emptyset$;
repeat
  Perform a single PSO iteration;
  if $f(S.\hat{\mathbf{y}}) \leq \epsilon$ **then**
    Isolate $S.\hat{\mathbf{y}}$;
    Perform a local search around $S.\hat{\mathbf{y}}$;
    if *a minimizer* $\mathbf{x}^*_{\mathcal{N}}$ *is found by the local search* **then**
      $\mathcal{X} \leftarrow \mathcal{X} \cup \{\mathbf{x}^*_{\mathcal{N}}\}$;
      Let $f(\mathbf{x}) \leftarrow H(\mathbf{x})$;
    end
  end
  Reinitialize the swarm S;
until *stopping condition is true*;
Return $\mathcal{X}$ as the set of multiple solutions;



: Effect of Sequential Niching for One Dimension

Parallel niching PSO

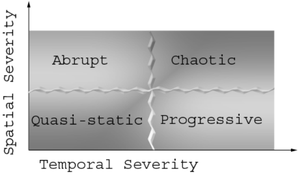
Create and initialize a n_x-dimensional *main* swarm, S;
repeat
  Train main swarm, S, for one iteration using *cognition-only* model;
  Update the fitness of each main swarm particle, $S.\mathbf{x}_i$;
  for *each sub-swarm S_k* **do**
    Train sub-swarm particles, $S_k.\mathbf{x}_i$, using a full model PSO;
    Update each particle's fitness;
    Update the swarm radius $S_k.R$;
  endFor
  If possible, merge sub-swarms;
  Allow sub-swarms to absorb any particles from the main swarm that moved into the sub-swarm;
  If possible, create new sub-swarms;
until *stopping condition is true*;
Return $S_k.\hat{\mathbf{y}}$ for each sub-swarm S_k as a solution;

- Objective: To find and track solutions in dynamically changing search spaces

$$\mathbf{x}^*(t) = \min_{\mathbf{x}} f(\mathbf{x}, \varpi(t))$$

where $\mathbf{x}^*(t)$ is the optimum found at time step t, and $\varpi(t)$ is a vector of time-dependent objective function control parameters

- Environment types:
  - Location of optima may change
  - Value of optima may change
  - Optima may disappear and new ones appear
  - Change frequencey
  - Change severity



: Environment Classes

Can PSO be applied to track an optimum?

- Only for quasi-static environments, to some success

What are the problems?

- Loss of diversity
- Memory
- Change detection

What can be done to address these problems?

- Diversity
  - Inject diversity into the swarm, but how much, and how?
  - Maintain diversity
- Memory
  - Re-evaluate personal best and neighborhood best positions
- Change detection
  - Use sentry particles

Maintains diversity throughout the search process

- Some particles attract one another, and others repell one another
- Velocity changes to

$$v_{ij}(t+1) = wv_{ij}(t) + c_1 r_1(t)[y_{ij}(t) - x_{ij}(t)] + c_2 r_2(t)[\hat{y}_j(t) - x_{ij}(t)] + a_{ij}(t)$$

where $\mathbf{a}_i$ is the particle acceleration, determining the magnitude of inter-particle repulsion

$$\mathbf{a}_i(t) = \sum_{l=1, i \neq l}^{n_s} \mathbf{a}_{il}(t)$$

- The repulsion force between particles i and l is

$$\mathbf{a}_{il}(t) = \begin{cases} \left(\frac{Q_i Q_l}{d_{il}^3}\right)(\mathbf{x}_i(t) - \mathbf{x}_l(t)) & \text{if } R_c \leq d_{il} \leq R_p \\ 0 & \text{otherwise} \end{cases}$$

- Based on quantum model of an atom, where orbiting electrons are replaced by a quantum cloud which is a probability distribution governing the position of the electron
- Developed as a simplified and less expensive version of the charged PSO
- Swarm contains
  - neutral particles following standard PSO updates
  - charged, or quantum particles, randomly placed within a multi-dimensional sphere

$$\mathbf{x}_i(t+1) = \begin{cases} \mathbf{x}_i(t) + \mathbf{v}_i(t+1) & \text{if } Q_i = 0 \\ \mathbf{B}_{\hat{\mathbf{y}}}(r_{cloud}) & \text{if } Q_i \neq 0 \end{cases}$$

  - charged particles uniformly sampled within the sphere

Can use different distributions:

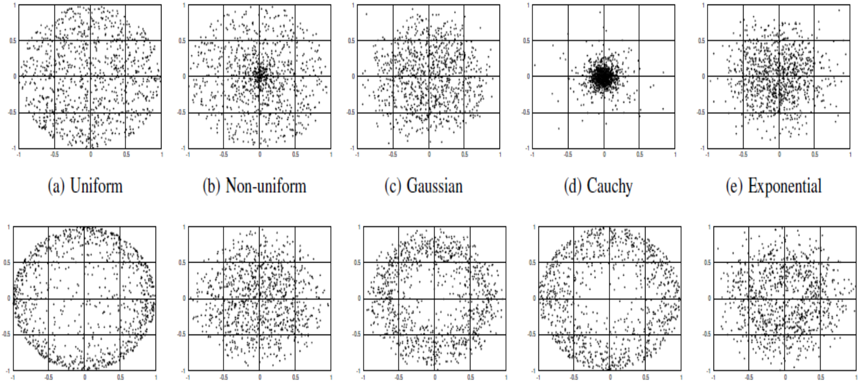
$$\mathbf{x}_i(t+1) \sim P(\hat{\mathbf{y}}(t), r_{cloud})$$

where P is some probability distribution and r_{cloud} is the quantum radius

Some alternative distributions to consider:

- Non-uniform (decreasing probability)
- Gaussian
- Cauchy
- Exponential
- Beta
- Triangular
- Weibull

Best distribution depends on type of dynamism



(a) Uniform (b) Non-uniform (c) Gaussian (d) Cauchy (e) Exponential

(f) Beta (g) Triangular-0 (h) Triangular-0.5 (i) Triangular-1 (j) Weibull

Constrained optimization problem:

$$\begin{aligned} \text{minimize} \quad & f(\mathbf{x}), \quad \mathbf{x} = (x_1, \ldots, x_{n_x}) \\ \text{subject to} \quad & g_m(\mathbf{x}) \leq 0, \; m = 1, \ldots, n_g \\ & h_m(\mathbf{x}) = 0, \; m = n_g + 1, \ldots, n_g + n_h \\ & x_j \in \text{dom}(x_j) \end{aligned}$$

where n_g and n_h are the number of inequality and equality constraints respectively

- How do we ensure that only feasible solutions are found?
- Boundary versus functional constraints
- For boundary constraints:
  - Do not allow particles that violate boundary constraints to become personal best positions
  - Reinitialize those elements that violate the boundary constraints within the bounds
- Reject infeasible solutions
  - Do not allow infeasible particles to become personal best or neighborhood best positions
  - Replace infeasible solutions with randomly generated, feasible solutions

Optimization problem is reformulated as

$$\text{minimize} \quad F(\mathbf{x}, t) = f(\mathbf{x}, t) + \lambda p(\mathbf{x}, t)$$

λ is the penalty coefficien, and $p(\mathbf{x}, t)$ is the (possibly) time-dependent penalty function

- How to find the best penalty coefficients?
- And the penalty?

$$p(\mathbf{x}_i, t) = \sum_{m=1}^{n_g+n_h} \lambda_m(t) p_m(\mathbf{x}_i)$$

where

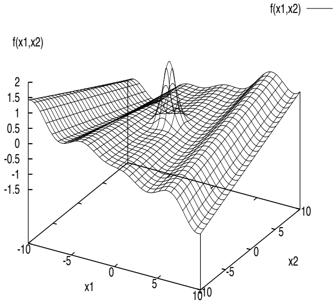
$$p_m(\mathbf{x}_i) = \begin{cases} \max\{0, g_m(\mathbf{x}_i)^\alpha\} & \text{if } m \in [1, \ldots, n_g] \\ |h_m(\mathbf{x}_i)|^\alpha & \text{if } m \in [n_g + 1, \ldots, n_g + n_h] \end{cases}$$

α is a positive constant, representing the power of the penalty

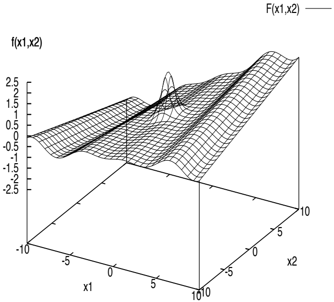
Constrained Optimization Problems

Penalty Methods (cont)

$$f(x_1, x_2) = \frac{x_1 \cos(x_1)}{20} + 2e^{-x_1^2-(x_2-1)^2} + 0.01x_1x_2$$



(a) Original function



(b) With penalty $p(x_1, x_2) = 3x_1$ and $\lambda = 0.05$

Constrained problem 1: Minimize the function

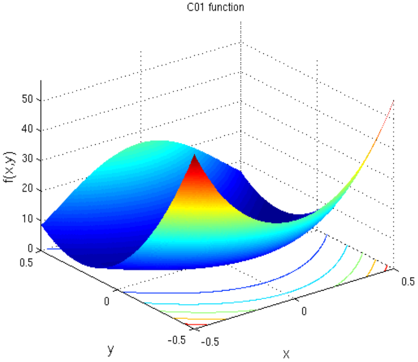
$$f(\mathbf{x}) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$$

subject to the nonlinear constraints,

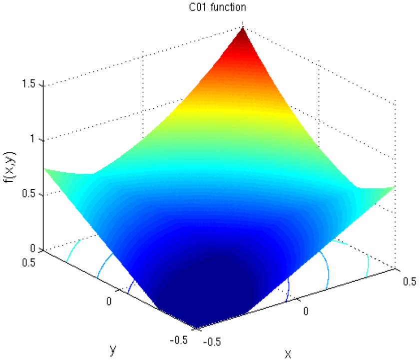
$$x_1 + x_2^2 \geq 0$$
$$x_1^2 + x_2 \geq 0$$

with $x_1 \in [-0.5, 0.5]$ and $x_2 \leq 1.0$.

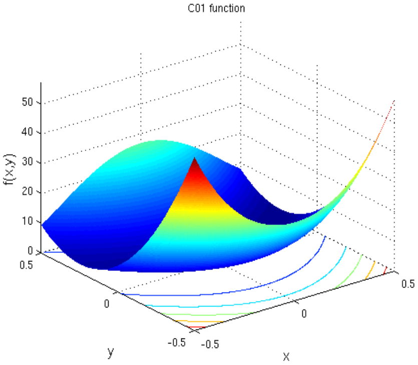
The global optimum is $\mathbf{x}^* = (0.5, 0.25)$, with $f(\mathbf{x}^*) = 0.25$



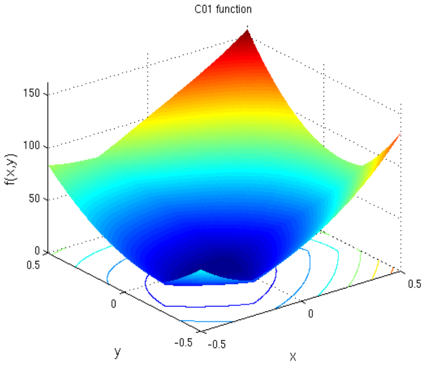
: Function Landscape



: Violation Space



: With $\lambda = 1$



: With $\lambda = 100$

Convert the constrained (primal) problem to an unconstrained problem by defining the Lagrangian for the constrained problem:

$$L(\mathbf{x}, \lambda_g, \lambda_h) = f(\mathbf{x}) + \sum_{m=1}^{n_g} \lambda_{gm} g_m(\mathbf{x}) + \sum_{m=n_g+1}^{n_g+n_h} \lambda_{hm} h_m(\mathbf{x})$$

Then maximize the Lagrangian (dual problem):

$$\begin{array}{ll} \text{maximize}_{\lambda_g, \lambda_h} & L(\mathbf{x}, \lambda_g, \lambda_h) \\ \text{subject to} & \lambda_{gm} \geq 0, \quad m = 1, \ldots, n_g + n_h \end{array}$$

The vector $\mathbf{x}^*$ that solves the primal problem, as well as the Lagrange multiplier vectors, λ_g^* and λ_h^*, can be found by solving the min-max problem,

$$\min_{\mathbf{x}} \max_{\lambda_g, \lambda_h} L(\mathbf{x}, \lambda_g, \lambda_h)$$

A coevolutionary PSO approach to solve the above min-max problem uses two swarms

- Swarm S_1 uses fitness function

$$f(\mathbf{x}) = \max_{\lambda_g, \lambda_h \in S_2} L(\mathbf{x}, \lambda_g, \lambda_h)$$

- Swarm S_2 uses fitness function

$$f(\lambda_g, \lambda_h) = \min_{\mathbf{x} \in S_1} L(\mathbf{x}, \lambda_g, \lambda_h)$$

Create and initialize two swarms, S_1 and S_2, where S_1 is n_x-dimensional and S_2 is $n_g + n_h$ dimensional;

repeat

  Run a PSO algorithm on swarm S_1 for $S_1.n_t$ iterations;

  Re-evaluate $S_2.\mathbf{y}_i(t), \forall i = 1, \ldots, S_2.n_s$;

  Run a PSO algorithm on swarm S_2 for $S_2.n_t$ iterations;

  Re-evaluate $S_1.\mathbf{y}_i(t), \forall i = 1, \ldots, S_1.n_s$;

until *stopping condition is true*;

Reformulate as a boundary constrained multi-objective optimization problem:

$$\mathbf{f}(\mathbf{x}) = (f(\mathbf{x}), p(\mathbf{x}))$$

Solve using any multi-objective PSO algorithm

Multi-objective problem:

$$\begin{array}{ll} \text{minimize} & \mathbf{f}(\mathbf{x}) \\ \text{subject to} & g_m(\mathbf{x}) \leq 0, \quad m = 1, \ldots, n_g \\ & h_m(\mathbf{x}) = 0, \quad m = n_g + 1, \ldots, n_g + n_h \\ & \mathbf{x} \in [\mathbf{x}_{min}, \mathbf{x}_{max}]^{n_x} \end{array}$$

where $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_{n_k}(\mathbf{x})) \in \mathcal{O} \subseteq \mathbb{R}^{n_k}$

$\mathcal{O}$ is referred to as the *objective space*

The search space, $\mathcal{S}$, is also referred to as the *decision space*

Important things to note:

- Goals are in conflict with one another
- Need to achieve a balance between these objectives
- A balance is achieved when a solution cannot improve any objective without degrading one or more of the other objectives
- There is not just one solution
- Solutions are referred to as *non-dominated solutions*
- Set of solutions is referred to as the Pareto-optimal set, and the corresponding objective vectors are referred to as the Pareto front

Definition:

$$
\begin{array}{ll}
\text{minimize} & \sum_{k=1}^{n_k} \omega_k f_k(\mathbf{x}) \\
\text{subject to} & g_m(\mathbf{x}) \leq 0, \quad m = 1, \ldots, n_g \\
& h_m(\mathbf{x}) = 0, \quad m = n_g + 1, \ldots, n_g + n_h \\
& \mathbf{x} \in [\mathbf{x}_{min}, \mathbf{x}_{max}]^{n_x} \\
& \omega_k \geq 0, k = 1, \ldots, n_k
\end{array}
$$

It is also usually assumed that $\sum_{k=1}^{n_k} \omega_k = 1$

Aggregation methods have the following problems:

- The algorithm has to be applied repeatedly to find different solutions if a single-solution algorithm is used
- It is difficult to get the best weight values, ω_k, since these are problem-dependent
- Aggregation methods can only be applied to generate members of the Pareto-optimal set when the Pareto front is concave, regardless of the values of ω_k

Domination: *A decision vector,* $\mathbf{x}_1$ *dominates a decision vector,* $\mathbf{x}_2$ *(denoted by* $\mathbf{x}_1 \prec \mathbf{x}_2$*), if and only if*

- $\mathbf{x}_1$ *is not worse than* $\mathbf{x}_2$ *in all objectives, i.e.* $f_k(\mathbf{x}_1) \leq f_k(\mathbf{x}_2), \forall k = 1, \ldots, n_k$*, and*
- $\mathbf{x}_1$ *is strictly better than* $\mathbf{x}_2$ *in at least one objective, i.e.* $\exists k = 1, \ldots, n_k : f_k(\mathbf{x}_1) < f_k(\mathbf{x}_2)$*.*

So, solution $\mathbf{x}_1$ is better than solution $\mathbf{x}_2$ if $\mathbf{x}_1 \prec \mathbf{x}_2$ (i.e. $\mathbf{x}_1$ dominates $\mathbf{x}_2$), which happens when $\mathbf{f}_1 \prec \mathbf{f}_2$

Pareto-optimal: *A decision vector,* $\mathbf{x}^* \in \mathcal{F}$ *is Pareto-optimal if there does not exist a decision vector,* $\mathbf{x} \neq \mathbf{x}^* \in \mathcal{F}$ *that dominates it. That is,* $\nexists k : f_k(\mathbf{x}) < f_k(\mathbf{x}^*)$. *An objective vector,* $\mathbf{f}^*(\mathbf{x})$, *is Pareto-optimal if* $\mathbf{x}$ *is Pareto-optimal.*

Pareto-optimal set: *The set of all Pareto-optimal decision vectors form the Pareto-optimal set,* $\mathcal{P}^*$. *That is,*

$$\mathcal{P}^* = \{\mathbf{x}^* \in \mathcal{F} | \nexists \mathbf{x} \in \mathcal{F} : \mathbf{x} \prec \mathbf{x}^*\}$$

Pareto-optimal front: *Given the objective vector,* $\mathbf{f}(\mathbf{x})$, *and the Pareto-optimal solution set,* $\mathcal{P}^*$, *then the Pareto-optimal front,* $\mathcal{PF}^* \subseteq \mathcal{O}$, *is defined as*

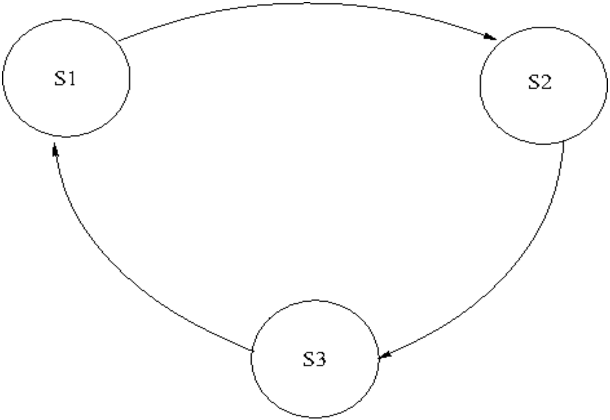
$$\mathcal{PF}^* = \{\mathbf{f} = (f_1(\mathbf{x}^*), f_2(\mathbf{x}^*), \ldots, f_k(\mathbf{x}^*)) | \mathbf{x}^* \in \mathcal{P}\}$$

A multi-swarm approach:

- Assume K sub-objectives
- K sub-swarms are used, where each optimizes one of the objectives
- Need a knowledge transfer strategy (KTS) to transfer information about best positions between sub-swarms
- Exchanged information are via selection of global guides, replacing the global best positions in the velocity updates
- Standard KTS: the ring KTS
  - Sub-swarms are arranged in a ring topology
  - Global guide of swarm S_k is swarm $S_{(k+1) \mod K}$

Multi-Objective Problems

VEPSO (cont)



Assume two objectives

$$\begin{aligned} S_1.v_{ij}(t+1) &= wS_1.v_{ij}(t) + c_1 r_{1j}(t)(S_1.y_{ij}(t) - S_1.x_{ij}(t)) \\ &+ c_2 r_{2j}(t)(S_2.\hat{y}_i(t) - S_1.x_{ij}(t)) \\ S_2.v_{ij}(t+1) &= wS_2.v_{ij}(t) + c_1 r_{1j}(t)(S_2.y_{ij}(t) - S_2.x_{ij}(t)) \\ &+ c_2 r_{ij}(t)(S_1.\hat{y}_j(t) - S.x_{2j}(t)) \end{aligned}$$

where sub-swarm S_1 evaluates individuals on the basis of objective $f_1(\mathbf{x})$, and sub-swarm S_2 uses objective $f_2(\mathbf{x})$

Local guide selection:

- Local guide replaces the personal best
- Update personal best position only if the new particle position dominates the previous personal best position

Global guide selection:

- Global guide replaces the neighborhood best
- Selection dictated by a knowledge transfer strategy (KTS):
    - Ring KTS
    - Random KTS

Using archives

- Objective of archive is to keep track of all non-dominated solutions
- Non-dominated solutions added to archive after each iteration
- Fixed-sized archives versus unlimited sizes
- Local versus global guides

Let $t = 0$;
Initialize the swarm, $S(t)$, and archive, $A(t)$;
repeat
  Evaluate $(S(t))$;
  $A(t+1) \leftarrow \text{Update}(S(t), A(t))$;
  $S(t+1) \leftarrow \text{Generate}(S(t), A(t))$;
  $t = t + 1$;
until *stopping condition is true*;

Curse of dimensionality:

- As dimensionality increases, performance deteriorates

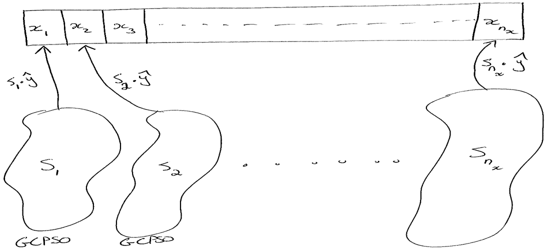
What to do?

- Increase number of particles
  - Increases computational complexity
  - Reduces step sizes, due to smaller difference vectors
- Reduce the complexity of the problem, using divide-and-conquer

Large Scale Optimization

Cooperative PSO (CPSO)

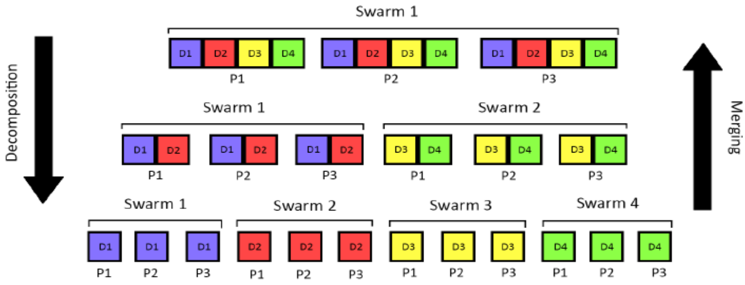
- Each particle is split into K separate parts of smaller dimension
- Each part is then optimized using a separate sub-swarm
- If $K = n_x$, each dimension is optimized by a separate sub-swarm
- What are the issues?
  - Problem if there are strong dependencies among variables
  - How should the fitness of sub-swarm particles be evaluated?



$K_1 = n_x \bmod K$ and $K_2 = K - (n_x \bmod K)$;
Initialize K_1 $\lceil n_x/K \rceil$-dimensional and K_2 $\lfloor n_x/K \rfloor$-dimensional swarms;
repeat
    for *each sub-swarm* $S_k, k = 1, \ldots, K$ **do**
        for *each particle* $i = 1, \ldots, S_k.n_s$ **do**
            if $f(\mathbf{b}(k, S_k.\mathbf{x}_i)) < f(\mathbf{b}(k, S_k.\mathbf{y}_i))$ **then**
                $S_k.\mathbf{y}_i = S_k.\mathbf{x}_i$;
            end
            if $f(\mathbf{b}(k, S_k.\mathbf{y}_i)) < f(\mathbf{b}(k, S_k.\hat{\mathbf{y}}))$ **then**
                $S_k.\hat{\mathbf{y}} = S_k.\mathbf{y}_i$;
            end
        end
        Apply velocity and position updates;
    end
until *stopping condition is true*;

How to cope with variable dependencies?

- Pre-processing to determine correlations and group correlated variables in same sub-swarm
- Random grouping
- Top-down versus bottom-up approaches



Summary

- Particle swarm optimization is an extremely simple, yet powerful optimization method
- Without changing the basic principles of PSO, minor modifications allow PSO to be applied to a wide range of problem classes, including:
    - Unconstrained
    - Constrained
    - Unimodal and multimodal
    - Continuous-valued and discrete-valued
    - Dynamically changing landscapes
    - Multi-objective
    - Multiple solutions
- Various combinations of the above problem types

Summary

- We can therefore safely say that PSO is a universal optimizer
- We do not say that PSO is the best for all classes of problems, and all landscape characteristics, only that it can be applied to solve a wide range of problem classes
- This tutorial is based on the content of the following reference: AP Engelbrecht, *Fundamentals of Computational Swarm Intelligence*, Wiley, 2005.