Particle Swarm Optimization: A Universal Optimizer?

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Presentation Outline I

1. Introduction
2. Optimization Problem Classes
3. Standard Particle Swarm Optimization
4. Discrete-valued Variables
5. Multi-Modal Optimization
6. Dynamic Optimization Problems
7. Constrained Optimization Problems
8. Multi-Objective Optimization Problems
9. Large Scale Optimization
Original PSO has been developed to solve optimization problems that are

- unconstrained/boundary constrained
- static
- single-objective
- continuous-valued

However:

- Can PSO be used to solve optimization problems of different classes, without significantly changing the principles of the basic PSO?
- If this is the case, we say that PSO is a universal optimizer
- For each problem class, what are the issues, and how can PSO be adapted to address these issues, while still maintaining the behavioral principles of PSO?
Goal:

- To show that PSO is a universal optimizer
- Not to present a review of the best possible approaches to solve optimization problems of the different problem classes, but to show that PSO can solve these problems
- Focus is on simple, efficient approaches
A number of different optimization problem classes can be identified:

- Unconstrained
- Boundary constrained
- Constrained
- Multi-objective, many-objective
- Multi-modal
- Dynamic and noisy
- Continuous-valued versus discrete-valued
- Large scale problems
Basic Foundations of Particle Swarm Optimization

Main Components

What are the main components?
- a swarm of particles
- each particle represents a candidate solution
- elements of a particle represent parameters to be optimized

The search process:
- Position updates

\[
x_i(t + 1) = x_i(t) + v_i(t + 1), \quad x_{ij}(0) \sim U(x_{\text{min},j}, x_{\text{max},j})
\]

- Velocity (step size)
  - drives the optimization process
  - step size
  - reflects experiential knowledge and socially exchanged information

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Social network structures are used to determine best positions/attractors

- Star Topology
- Ring Topology
- Von Neumann Topology
Basic Foundations of Particle Swarm Optimization

global best (gbest) PSO

- uses the star social network
- velocity update per dimension:

\[ v_{ij}(t+1) = v_{ij}(t) + c_1 r_{1j}(t) [y_{ij}(t) - x_{ij}(t)] + c_2 r_{2j}(t) [\hat{y}_j(t) - x_{ij}(t)] \]

- \( v_{ij}(0) = 0 \) (preferred)
- \( c_1, c_2 \) are positive acceleration coefficients
- \( r_{1j}(t), r_{2j}(t) \sim U(0, 1) \)
- note that a random number is sampled for each dimension
**gbest PSO (cont)**

- \( y_i(t) \) is the personal best position calculated as (assuming minimization)

  \[
  y_i(t + 1) = \begin{cases} 
  y_i(t) & \text{if } f(x_i(t + 1)) \geq f(y_i(t)) \\
  x_i(t + 1) & \text{if } f(x_i(t + 1)) < f(y_i(t))
  \end{cases}
  \]

- \( \hat{y}(t) \) is the global best position calculated as

  \[
  \hat{y}(t) \in \{y_0(t), \ldots, y_{n_s}(t)\} | f(\hat{y}(t)) = \min\{f(y_0(t)), \ldots, f(y_{n_s}(t))\}
  \]

  or (removing memory of best positions)

  \[
  \hat{y}(t) = \min\{f(x_0(t)), \ldots, f(x_{n_s}(t))\}
  \]

  where \( n_s \) is the number of particles in the swarm
Basic Foundations of Particle Swarm Optimization
local best (lbest) PSO

- uses the ring social network

\[ v_{ij}(t + 1) = v_{ij}(t) + c_1 r_{1j}(t)[y_{ij}(t) - x_{ij}(t)] + c_2 r_{2j}(t)[\hat{y}_{ij}(t) - x_{ij}(t)] \]

- \( \hat{y}_i \) is the neighborhood best, defined as

\[ \hat{y}_i(t + 1) \in \{N_i|f(\hat{y}_i(t + 1)) = \min\{f(x)\}, \forall x \in N_i\} \]

with the neighborhood defined as

\[ N_i = \{y_{i-n_{N_i}}(t), y_{i-n_{N_i}+1}(t), \ldots, y_i(t), y_{i+1}(t), \ldots, y_{i+n_{N_i}}(t)\} \]

where \( n_{N_i} \) is the neighborhood size

- neighborhoods based on particle indices, not spatial information
- neighborhoods overlap to facilitate information exchange
previous velocity, $v_i(t)$
- inertia component
- memory of previous flight direction
- prevents particle from drastically changing direction

cognitive component, $c_1 r_1 (y_i - x_i)$
- quantifies performance relative to past performances
- memory of previous best position
- nostalgia

social component, $c_2 r_2 (\hat{y}_i - x_i)$
- quantifies performance relative to neighbors
- envy
**Synchronous Iteration Strategy**

Create and initialize the swarm;
repeat

for each particle do
  Evaluate particle’s fitness;
  Update particle’s personal best position;
  Update particle’s neighborhood best position;
end

for each particle do
  Update particle’s velocity;
  Update particle’s position;
end

until stopping condition is true;

**Asynchronous Iteration Strategy**

Create and initialize the swarm;
repeat

for each particle do
  Update the particle’s velocity;
  Update the particle’s position;
  Evaluate particle’s fitness;
  Update the particle’s personal best position;
  Update the particle’s neighborhood best position;
end

until stopping condition is true;
Discrete-Valued Variables

Introduction

What is the problem?

- PSO was originally developed for optimizing continuous-valued variables
- That is $x_{ij} \in \mathbb{R}$
- Uses vector algebra on floating-point vectors to adjust search positions

How do we adapt PSO so that $x_{ij} \in \{0, 1\}$?
Discrete-Valued Variables

Binary PSO

Adapting PSO for binary-valued variables: Binary PSO

- Velocity remains a floating-point vector, but meaning changes
- Velocity is no longer a step size, but is used to determine a probability of selecting bit 0 or bit 1
- Position is a bit vector, i.e. \( x_{ij} \in \{0, 1\} \)
- How to interpret velocity as a probability?

\[
p_{ij}(t) = \frac{1}{1 + e^{-v_{ij}(t)}}
\]

- Then, position update changes to

\[
x_{ij}(t + 1) = \begin{cases} 
1 & \text{if } U(0, 1) < p_{ij}(t + 1) \\
0 & \text{otherwise}
\end{cases}
\]
Velocity clamping:

- sets the minimal probability for a bit change
- if $V_{\text{max},j} = 4$, then $\text{sig}(V_{\text{max},j}) = 0.982$ is the probability of $x_{ij}$ to change to bit 1, and 0.018 the probability to change to bit 0
- small values for $V_{\text{max},j}$ promotes exploration
- for $V_{\text{max},j} = 0$, the search changes to a random search
- large values for $V_{\text{max},j}$ promotes exploitation
- start with small $V_{\text{max},j}$ that increases over time
Inertia weight:

- $w < 1$ works against convergence, as $v_{ij}$ becomes zero over time, and each bit then has a 50% change of changing velocity should not become zero
- start with small $w$, increase over time

Velocity initialization: Initialize to zero.

- for $v_{ij} > 1$, $\lim_{t \to \infty} \text{sig}(v_{ij}(t)) \to 1$ and the probability that all bits change to 1 increases
- for $v_{ij} < -1$, $\lim_{t \to \infty} \text{sig}(v_{ij}(t)) \to 0$ and the probability that all bits change to 0 increases
Some issues with the binary PSO:

- Changes the meaning of the velocity update
  - No longer a step size
  - No longer a search trajectory
- Effect of control parameters change
- Theoretical analysis of standard PSO no longer applies
Discrete-Valued Variables
Angle Modulated PSO (AMPSO)

An approach to solve a $\mathbb{B}^{n_x}$-dimensional problem in $\mathbb{R}^4$

- Velocities and particle positions remain floating-point vectors
- Find a bitstring generating function, used to generate the bitstring solution
- The generating function:

$$g(x) = \sin(2\pi(x - a) \times b \times \cos(2\pi(x - a) \times c)) + d$$

where $x$ is a single element from a set of evenly separated intervals determined by the required number of bits that need to be generated
Discrete-Valued Variables

AMPSO (cont)

\[ g(x) = \sin(2\pi(x - a) \times b \times \cos(2\pi(x - a) \times c)) + d \]

The coefficients determine the shape of the generating function:

- **a**: horizontal shift of generating function
- **b**: maximum frequency of the sin function
- **c**: frequency of the cos function
- **d**: vertical shift of generating function
Use a standard PSO to find the best values for these coefficients
Generate a swarm of 4-dimensional particles;
repeat
  Apply any PSO for one iteration;
  for each particle do
    Substitute values for coefficients $a$, $b$, $c$ and $d$ into generating function;
    Produce $n_x$ bit-values to form a bit-vector solution;
    Calculate the fitness of the bit-vector solution in the original bit-valued space;
  end
until a convergence criterion is satisfied;
Assuming minimization,

**Boundary constrained optimization problem:**

\[
\text{minimize } \quad f(\mathbf{x}), \quad \mathbf{x} = (x_1, x_2, \ldots, x_{n_x}) \\
\text{subject to} \quad x_j \in \text{dom}(x_j)
\]

where \( \mathbf{x} \in \mathcal{F} = S \), and \( \text{dom}(x_j) \) is the domain of variable \( x_j \).

**Multi-solution problem:** Find a set of solutions,

\[
\mathcal{X} = \{ \mathbf{x}_1^*, \mathbf{x}_2^*, \ldots, \mathbf{x}_{n_X}^* \}
\]

such that each \( \mathbf{x}^* \in \mathcal{X} \) is a minimum of the general optimization problem
Niching capability of PSO:

- Can the *g*best PSO find more than one solution?
  - Formal proofs showed that all particles converge to a weighted average of their personal best and global best positions:

\[
\lim_{t \to \infty} x_i(t) = \frac{c_1 y_i + c_2 \hat{y}}{c_1 + c_2}
\]

- Therefore, only one solution can be found
- What if we re-run the algorithm? No guarantee to find different solutions

- What about *l*best PSO?
  - Neighborhoods may converge to different solutions
  - However, due to overlapping neighborhoods, particles are still attracted to one solution
  - Formal proof exist to show that all particles converge in the limit
Multi-Modal Optimization ( Niching)

Objection Function Stretching

Sequential niching, stretching the function to remove found minima

Create and initialize a \( n_x \)-dimensional swarm, \( S \), and \( \mathcal{X} = \emptyset \);

repeat

\begin{itemize}
  \item Perform a single PSO iteration;
  \item if \( f(S.\hat{y}) \leq \epsilon \) then
    \begin{itemize}
      \item Isolate \( S.\hat{y} \);
      \item Perform a local search around \( S.\hat{y} \);
      \item if a minimizer \( x^*_N \) is found by the local search then
        \begin{itemize}
          \item \( \mathcal{X} \leftarrow \mathcal{X} \cup \{x^*_N\} \);
          \item Let \( f(x) \leftarrow H(x) \);
        \end{itemize}
    \end{itemize}
  \item end
\end{itemize}

\item Reinitialize the swarm \( S \);

until stopping condition is true;

Return \( \mathcal{X} \) as the set of multiple solutions;
Effect of Sequential Niching for One Dimension
Multi-Modal Optimization (Niching)

Niche PSO

Parallel niching PSO

Create and initialize a $n_x$-dimensional main swarm, $S$;

repeat

Train main swarm, $S$, for one iteration using \textit{cognition-only} model;

Update the fitness of each main swarm particle, $S.x_i$;

for each sub-swarm $S_k$ do

Train sub-swarm particles, $S_k.x_i$, using a full model PSO;

Update each particle’s fitness;

Update the swarm radius $S_k.R$;

endFor

If possible, merge sub-swarms;

Allow sub-swarms to absorb any particles from the main swarm that moved into the sub-swarm;

If possible, create new sub-swarms;

until \textit{stopping condition is true};

Return $S_k.\hat{y}$ for each sub-swarm $S_k$ as a solution;
Objective: To find and track solutions in dynamically changing search spaces

\[ x^*(t) = \min_x f(x, \omega(t)) \]

where \( x^*(t) \) is the optimum found at time step \( t \), and \( \omega(t) \) is a vector of time-dependent objective function control parameters.

Environment types:
- Location of optima may change
- Value of optima may change
- Optima may disappear and new ones appear
- Change frequency
- Change severity

: Environment Classes
Can PSO be applied to track an optimum?
- Only for quasi-static environments, to some success

What are the problems?
- Loss of diversity
- Memory
- Change detection
What can be done to address these problems?

- **Diversity**
  - Inject diversity into the swarm, but how much, and how?
  - Maintain diversity

- **Memory**
  - Re-evaluate personal best and neighborhood best positions

- **Change detection**
  - Use sentry particles
Dynamic Optimization Problems

Charged PSO

Maintains diversity throughout the search process

- Some particles attract one another, and others repel one another
- Velocity changes to

\[ v_{ij}(t+1) = w v_{ij}(t) + c_1 r_1(t) [y_{ij}(t) - x_{ij}(t)] + c_2 r_2(t) [\hat{y}_j(t) - x_{ij}(t)] + a_{ij}(t) \]

where \( a_i \) is the particle acceleration, determining the magnitude of inter-particle repulsion

\[ a_i(t) = \sum_{l=1,l \neq i}^{n_s} a_{il}(t) \]

- The repulsion force between particles \( i \) and \( l \) is

\[ a_{il}(t) = \begin{cases} \left( \frac{Q_i Q_l}{d_{il}^3} \right) \left( x_i(t) - x_l(t) \right) & \text{if } R_c \leq d_{il} \leq R_p \cr 0 & \text{otherwise} \end{cases} \]
Dynamic Optimization Problems
Quantum PSO

- Based on quantum model of an atom, where orbiting electrons are replaced by a quantum cloud which is a probability distribution governing the position of the electron
- Developed as a simplified and less expensive version of the charged PSO
- Swarm contains
  - neutral particles following standard PSO updates
  - charged, or quantum particles, randomly placed within a multi-dimensional sphere

\[
x_i(t + 1) = \begin{cases} 
  x_i(t) + v_i(t + 1) & \text{if } Q_i = 0 \\
  B \hat{y}(r_{cloud}) & \text{if } Q_i \neq 0
\end{cases}
\]

- charged particles uniformly sampled within the sphere
Can use different distributions:

\[ x_i(t + 1) \sim P(\hat{y}(t), r_{\text{cloud}}) \]

where \( P \) is some probability distribution and \( r_{\text{cloud}} \) is the quantum radius

Some alternative distributions to consider:
- Non-uniform (decreasing probability)
- Gaussian
- Cauchy
- Exponential
- Beta
- Triangular
- Weibull

Best distribution depends on type of dynamism
Dynamic Optimization Problems
Quantum PSO (cont)

(a) Uniform
(b) Non-uniform
(c) Gaussian
(d) Cauchy
(e) Exponential

(f) Beta
(g) Triangular-0
(h) Triangular-0.5
(i) Triangular-1
(j) Weibull
Constrained optimization problem:

\[
\begin{align*}
\text{minimize} \quad f(x), \quad & x = (x_1, \ldots, x_{n_x}) \\
\text{subject to} \quad & g_m(x) \leq 0, \ m = 1, \ldots, n_g \\
& h_m(x) = 0, \ m = n_g + 1, \ldots, n_g + n_h \\
& x_j \in \text{dom}(x_j)
\end{align*}
\]

where \( n_g \) and \( n_h \) are the number of inequality and equality constraints respectively.
How do we ensure that only feasible solutions are found?

Boundary versus functional constraints

For boundary constraints:
- Do not allow particles that violate boundary constraints to become personal best positions
- Reinitialize those elements that violate the boundary constraints within the bounds

Reject infeasible solutions
- Do not allow infeasible particles to become personal best or neighborhood best positions
- Replace infeasible solutions with randomly generated, feasible solutions
Optimization problem is reformulated as

\[
\text{minimize} \quad F(x, t) = f(x, t) + \lambda p(x, t)
\]

\(\lambda\) is the penalty coefficient, and \(p(x, t)\) is the (possibly) time-dependent penalty function

- How to find the best penalty coefficients?
- And the penalty?

\[
p(x_i, t) = \sum_{m=1}^{ng+nh} \lambda_m(t) p_m(x_i)
\]

where

\[
p_m(x_i) = \begin{cases} 
\max\{0, g_m(x_i)^\alpha\} & \text{if } m \in [1, \ldots, ng] \\
|h_m(x_i)|^\alpha & \text{if } m \in [ng + 1, \ldots, ng + nh]
\end{cases}
\]

\(\alpha\) is a positive constant, representing the power of the penalty.
Constrained Optimization Problems
Penalty Methods (cont)

\[ f(x_1, x_2) = \frac{x_1 \cos(x_1)}{20} + 2e^{-x_1^2-(x_2-1)^2} + 0.01x_1x_2 \]

(a) Original function
(b) With penalty \( p(x_1, x_2) = 3x_1 \) and \( \lambda = 0.05 \)
Constrained problem 1: Minimize the function

\[ f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2 \]

subject to the nonlinear constraints,

\[
\begin{align*}
  x_1 + x_2^2 &\geq 0 \\
  x_2^2 + x_1 &\geq 0 
\end{align*}
\]

with \( x_1 \in [-0.5, 0.5] \) and \( x_2 \leq 1.0 \).

The global optimum is \( x^* = (0.5, 0.25) \), with \( f(x^*) = 0.25 \).
Constrained Optimization Problems
Penalty Methods (cont)

: Function Landscape

: Violation Space
Constrained Optimization Problems
Penalty Methods (cont)

: With $\lambda = 1$

: With $\lambda = 100$
Convert the constrained (primal) problem to an unconstrained problem by defining the Lagrangian for the constrained problem:

\[ L(x, \lambda_g, \lambda_h) = f(x) + \sum_{m=1}^{n_g} \lambda_g g_m(x) + \sum_{m=n_g+1}^{n_g+n_h} \lambda_h h_m(x) \]

Then maximize the Lagrangian (dual problem):

\[
\begin{align*}
\text{maximize}_{\lambda_g, \lambda_h} & \quad L(x, \lambda_g, \lambda_h) \\
\text{subject to} & \quad \lambda_{gm} \geq 0, \quad m = 1, \ldots, n_g + n_h
\end{align*}
\]
The vector $\mathbf{x}^*$ that solves the primal problem, as well as the Lagrange multiplier vectors, $\lambda_g^*$ and $\lambda_h^*$, can be found by solving the min-max problem,

$$\min_{\mathbf{x}} \max_{\lambda_g, \lambda_h} L(\mathbf{x}, \lambda_g, \lambda_h)$$
A coevolutionary PSO approach to solve the above min-max problem uses two swarms

- **Swarm** $S_1$ uses fitness function
  
  $$f(x) = \max_{\lambda_g, \lambda_h \in S_2} L(x, \lambda_g, \lambda_h)$$

- **Swarm** $S_2$ uses fitness function
  
  $$f(\lambda_g, \lambda_h) = \min_{x \in S_1} L(x, \lambda_g, \lambda_h)$$
Create and initialize two swarms, $S_1$ and $S_2$, where $S_1$ is $n_x$-dimensional and $S_2$ is $n_g + n_h$ dimensional;

repeat
  Run a PSO algorithm on swarm $S_1$ for $S_1.n_t$ iterations;
  Re-evaluate $S_2.y_i(t), \forall i = 1, \ldots, S_2.n_s$;
  Run a PSO algorithm on swarm $S_2$ for $S_2.n_t$ iterations;
  Re-evaluate $S_1.y_i(t), \forall i = 1, \ldots, S_1.n_s$;
until stopping condition is true;
Reformulate as a boundary constrained multi-objective optimization problem:

\[
f(x) = (f(x), p(x))
\]

Solve using any multi-objective PSO algorithm.
Multi-objective problem:

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad g_m(x) \leq 0, \quad m = 1, \ldots, n_g \\
& \quad h_m(x) = 0, \quad m = n_g + 1, \ldots, n_g + n_h \\
& \quad x \in [x_{\text{min}}, x_{\text{max}}]^n \\
\end{align*}
\]

where \( f(x) = (f_1(x), f_2(x), \ldots, f_{n_k}(x)) \in \mathcal{O} \subseteq \mathbb{R}^{n_k} \)

\( \mathcal{O} \) is referred to as the *objective space*

The search space, \( S \), is also referred to as the *decision space*
Important things to note:

- Goals are in conflict with one another
- Need to achieve a balance between these objectives
- A balance is achieved when a solution cannot improve any objective without degrading one or more of the other objectives
- There is not just one solution
- Solutions are referred to as non-dominated solutions
- Set of solutions is referred to as the Pareto-optimal set, and the corresponding objective vectors are referred to as the Pareto front
Multi-Objective Problems
Weighted Aggregation

Definition:

\[
\begin{align*}
\text{minimize} & \quad \sum_{k=1}^{n_k} \omega_k f_k(x) \\
\text{subject to} & \quad g_m(x) \leq 0, \quad m = 1, \ldots, n_g \\
& \quad h_m(x) = 0, \quad m = n_g + 1, \ldots, n_g + n_h \\
& \quad x \in [x_{min}, x_{max}]^{n_x} \\
& \quad \omega_k \geq 0, k = 1, \ldots, n_k
\end{align*}
\]

It is also usually assumed that \(\sum_{k=1}^{n_k} \omega_k = 1\)
Aggregation methods have the following problems:

- The algorithm has to be applied repeatedly to find different solutions if a single-solution algorithm is used.
- It is difficult to get the best weight values, $\omega_k$, since these are problem-dependent.
- Aggregation methods can only be applied to generate members of the Pareto-optimal set when the Pareto front is concave, regardless of the values of $\omega_k$. 

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**Domination:** A decision vector, \( \mathbf{x}_1 \) dominates a decision vector, \( \mathbf{x}_2 \) (denoted by \( \mathbf{x}_1 \preceq \mathbf{x}_2 \)), if and only if

- \( \mathbf{x}_1 \) is not worse than \( \mathbf{x}_2 \) in all objectives, i.e.
  \[ f_k(\mathbf{x}_1) \leq f_k(\mathbf{x}_2), \forall k = 1, \ldots, n_k, \text{ and} \]

- \( \mathbf{x}_1 \) is strictly better than \( \mathbf{x}_2 \) in at least one objective, i.e.
  \[ \exists k = 1, \ldots, n_k : f_k(\mathbf{x}_1) < f_k(\mathbf{x}_2). \]

So, solution \( \mathbf{x}_1 \) is better than solution \( \mathbf{x}_2 \) if \( \mathbf{x}_1 \preceq \mathbf{x}_2 \) (i.e. \( \mathbf{x}_1 \) dominates \( \mathbf{x}_2 \)), which happens when \( f_1 \preceq f_2 \).
Multi-Objective Problems
Pareto-Optimality (cont)

**Pareto-optimal**: A decision vector, \( x^* \in \mathcal{F} \) is Pareto-optimal if there does not exist a decision vector, \( x \neq x^* \in \mathcal{F} \) that dominates it. That is, \( \exists k : f_k(x) < f_k(x^*) \). An objective vector, \( f^*(x) \), is Pareto-optimal if \( x \) is Pareto-optimal.

**Pareto-optimal set**: The set of all Pareto-optimal decision vectors form the Pareto-optimal set, \( \mathcal{P}^* \). That is,

\[
\mathcal{P}^* = \{ x^* \in \mathcal{F} | \forall x \in \mathcal{F} : x \preceq x^* \}
\]

**Pareto-optimal front**: Given the objective vector, \( f(x) \), and the Pareto-optimal solution set, \( \mathcal{P}^* \), then the Pareto-optimal front, \( \mathcal{P}\mathcal{F}^* \subseteq \mathcal{O} \), is defined as

\[
\mathcal{P}\mathcal{F}^* = \{ f = (f_1(x^*), f_2(x^*), \ldots, f_k(x^*)) | x^* \in \mathcal{P} \}
\]
A multi-swarm approach:

- Assume $K$ sub-objectives
- $K$ sub-swarms are used, where each optimizes one of the objectives
- Need a knowledge transfer strategy (KTS) to transfer information about best positions between sub-swarms
- Exchanged information are via selection of global guides, replacing the global best positions in the velocity updates
- Standard KTS: the ring KTS
  - Sub-swarms are arranged in a ring topology
  - Global guide of swarm $S_k$ is swarm $S_{(k+1) \mod K}$
Multi-Objective Problems

VEPSO (cont)

![Diagram showing connections between S1, S2, and S3]
Assume two objectives

\[
S_1. v_{ij}(t + 1) = wS_1. v_{ij}(t) + c_1 r_{1j}(t)(S_1.y_{ij}(t) - S_1.x_{ij}(t)) \\
+ c_2 r_{2j}(t)(S_2.\hat{y}_i(t) - S_1.x_{ij}(t)) \\
S_2. v_{ij}(t + 1) = wS_2. v_{ij}(t) + c_1 r_{1j}(t)(S_2.y_{ij}(t) - S_2.x_{ij}(t)) \\
+ c_2 r_{ij}(t)(S_1.\hat{y}_j(t) - S.x_{2j}(t))
\]

where sub-swarm \( S_1 \) evaluates individuals on the basis of objective \( f_1(x) \), and sub-swarm \( S_2 \) uses objective \( f_2(x) \)
Multi-Objective Problems

VEPSO (cont)

Local guide selection:
- Local guide replaces the personal best
- Update personal best position only if the new particle position dominates the previous personal best position

Global guide selection:
- Global guide replaces the neighborhood best
- Selection dictated by a knowledge transfer strategy (KTS):
  - Ring KTS
  - Random KTS
Using archives

- Objective of archive is to keep track of all non-dominated solutions
- Non-dominated solutions added to archive after each iteration
- Fixed-sized archives versus unlimited sizes
- Local versus global guides

Let $t = 0$;
Initialize the swarm, $S(t)$, and archive, $A(t)$;

repeat

Evaluate $(S(t))$;
$A(t + 1) \leftarrow \text{Update}(S(t), A(t))$;
$S(t + 1) \leftarrow \text{Generate}(S(t), A(t))$;
$t = t + 1$;

until stopping condition is true;
Curse of dimensionality:
- As dimensionality increases, performance deteriorates

What to do?
- Increase number of particles
  - Increases computational complexity
  - Reduces step sizes, due to smaller difference vectors
- Reduce the complexity of the problem, using divide-and-conquer
Large Scale Optimization

Cooperative PSO (CPSO)

- Each particle is split into $K$ separate parts of smaller dimension
- Each part is then optimized using a separate sub-swarm
- If $K = n_x$, each dimension is optimized by a separate sub-swarm
- What are the issues?
  - Problem if there are strong dependencies among variables
  - How should the fitness of sub-swarm particles be evaluated?
\( K_1 = n_x \mod K \) and \( K_2 = K - (n_x \mod K) \);
Initialize \( K_1 \left[ \frac{n_x}{K} \right] \)-dimensional and \( K_2 \left[ \frac{n_x}{K} \right] \)-dimensional swarms;

repeat
  for each sub-swarm \( S_k, k = 1, \ldots, K \) do
    for each particle \( i = 1, \ldots, S_k.n_s \) do
      if \( f(b(k, S_k.x_i)) < f(b(k, S_k.y_i)) \) then
        \( S_k.y_i = S_k.x_i \);
      end
      if \( f(b(k, S_k.y_i)) < f(b(k, S_k.\hat{y})) \) then
        \( S_k.\hat{y} = S_k.y_i \);
      end
    end
    Apply velocity and position updates;
  end
until stopping condition is true;
Large Scale Optimization

CPSO (cont)

How to cope with variable dependencies?

- Pre-processing to determine correlations and group correlated variables in same sub-swarm
- Random grouping
- Top-down versus bottom-up approaches
Summary

• Particle swarm optimization is an extremely simple, yet powerful optimization method
• Without changing the basic principles of PSO, minor modifications allow PSO to be applied to a wide range of problem classes, including:
  • Unconstrained
  • Constrained
  • Unimodal and multimodal
  • Continuous-valued and discrete-valued
  • Dynamically changing landscapes
  • Multi-objective
  • Multiple solutions
• Various combinations of the above problem types
We can therefore safely say that PSO is a universal optimizer.

We do not say that PSO is the best for all classes of problems, and all landscape characteristics, only that it can be applied to solve a wide range of problem classes.

This tutorial is based on the content of the following reference: AP Engelbrecht, *Fundamentals of Computational Swarm Intelligence*, Wiley, 2005.